Indentation load–displacement relations at the elastic deformation stage

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Nanoindentation experiments are widely used by the materials science community in the investigation of Young's modulus and the hardness of thin films. In this paper, the problem is modeled as normal indentation of an elastic half-space by a rigid frictionless axisymmetric power-law indenter. An analytical solution, which relates the indentation load to the penetration depth and includes complete contact and incomplete contact situations, is presented. The solution is valid as long as the contact area is simply connected. A domain is simply connected if any simple closed curve can be shrunk to a point continuously in the set.

When an axisymmetric punch indents normally into a half-space (i.e., $-\infty < x < \infty$, $-\infty < y < \infty$, and $z > 0$), there are two possibilities: one is that the whole punch surface contacts with the half-space; the other is that only part of the punch contacts with the half-space. Following the terminology by Gladwell [1], the first contact is called complete, and the second one is termed incomplete. In the second case (Fig. 1a), the contact pressure will drop to zero at the boundary of the contact region. Complete contact can be further classified into critical complete contact (Fig. 1b) and general complete contact (Fig. 1c). In general complete contact, pressure at the punch edge goes to infinity; in critical complete contact, pressure drops to zero at the punch edge and the pressure profile is similar to that of incomplete contact.

In this paper, we consider a rigid frictionless axisymmetric power-law indenter with its axis of revolution as the z-axis indenting normally into the plane $z = 0$ of an elastic half-space $z \geq 0$. The problem is considered using the linear theory of elasticity and the half-space is assumed to be isotropic and homogeneous. The contact region between the indenter and the half-space is simply connected. The following equations give the relevant displacement and stresses for the half-space. The vertical component of the displacement is denoted by u_z, and the stress components have two subscripts corresponding to the appropriate coordinates. E and ν are Young's modulus and Poisson's ratio of the half-space, respectively.

As Fig. 1 shows, the boundary conditions for the half-space at $z = 0$ are

$$\tau_{zr} = \tau_{z\theta} = 0 \qquad (0 \leq r < \infty) \tag{1}$$

$$\sigma_{zz} = 0 \qquad (r > b) \tag{2}$$

$$u_z = h + a_\alpha r^\alpha \qquad (0 \leq r \leq b) \tag{3}$$

where α is a positive real number and h is the indentation depth. The second term on the right-hand side of Equation 3 describes the indenter shape. The radius of the contact area (b) is less than or equal to the punch radius (a).

For incomplete contact (Fig. 1a), the following condition has to be satisfied [2]:

$$\frac{2}{\sqrt{\pi}}h + (1+\alpha)\cdot a_\alpha \cdot \frac{\Gamma\left(\frac{2+\alpha}{2}\right)}{\Gamma\left(\frac{3+\alpha}{2}\right)} a^\alpha < 0. \tag{4}$$

The radius of the contact area and the indentation depth are related by the following equation:

$$b = \left[-\frac{2}{\sqrt{\pi}}\cdot\frac{1}{1+\alpha}\cdot\frac{\Gamma\left(\frac{3+\alpha}{2}\right)}{\Gamma\left(\frac{2+\alpha}{2}\right)}\cdot\frac{1}{a_\alpha}\right]^{\frac{1}{\alpha}} h^{\frac{1}{\alpha}}. \tag{5}$$

Putting Equation 5 into the load–displacement relation [2], we have

$$P = 2\left[\frac{2}{\sqrt{\pi}}\right]^{\frac{1}{\alpha}} \frac{\alpha}{(1+\alpha)^{\frac{1+\alpha}{\alpha}}}\cdot\left[\frac{\Gamma\left(\frac{3+\alpha}{2}\right)}{\Gamma\left(\frac{2+\alpha}{2}\right)}\right]^{\frac{1}{\alpha}} \cdot\frac{1}{(-a_\alpha)^{\frac{1}{\alpha}}}\cdot\frac{E}{1-\nu^2}\cdot h^{\frac{1+\alpha}{\alpha}}. \tag{6}$$

Equation 6 shows that for a power-law indenter, its load–displacement follows a power-law relationship for incomplete contact situations.

Critical complete contact (Fig. 1b) is a transition point from incomplete contact to general complete contact; and the following condition has to be satisfied:

$$\frac{2}{\sqrt{\pi}}h + (1+\alpha)\cdot a_\alpha \cdot \frac{\Gamma\left(\frac{2+\alpha}{2}\right)}{\Gamma\left(\frac{3+\alpha}{2}\right)} a^\alpha = 0. \tag{7}$$

The indentation depth is

$$h = -\frac{\sqrt{\pi}}{2}(1+\alpha)\cdot a_\alpha \cdot \frac{\Gamma\left(\frac{2+\alpha}{2}\right)}{\Gamma\left(\frac{3+\alpha}{2}\right)}\cdot a^\alpha \tag{8}$$

and the corresponding load is

$$P = -\sqrt{\pi}\alpha\cdot\frac{\Gamma\left(\frac{2+\alpha}{2}\right)}{\Gamma\left(\frac{3+\alpha}{2}\right)}\cdot a_\alpha\cdot a^{1+\alpha}\frac{E}{1-\nu^2}. \tag{9}$$

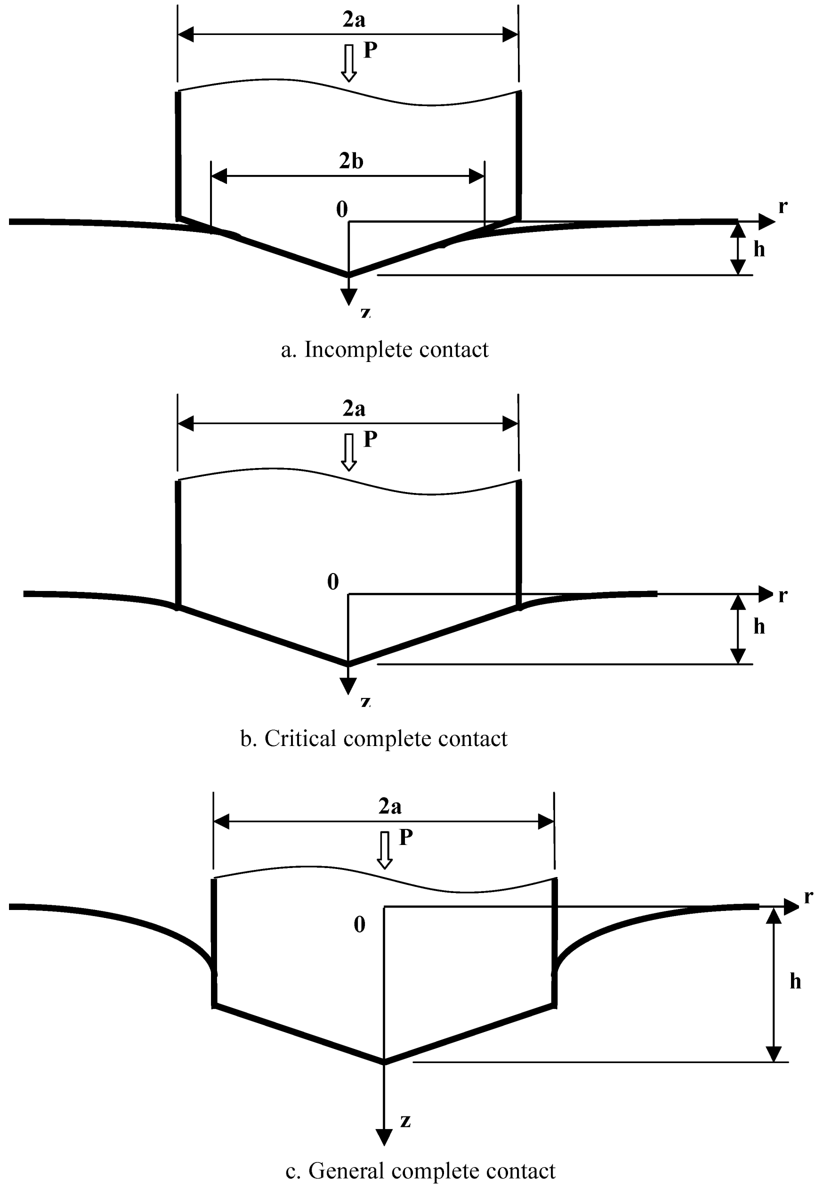

Figure 1 Different contacts.

For complete contact (Fig. 1c), the following relation holds:

$$\frac{2}{\sqrt{\pi}}h + (1+\alpha)\cdot a_\alpha \cdot \frac{\Gamma\left(\frac{2+\alpha}{2}\right)}{\Gamma\left(\frac{3+\alpha}{2}\right)}a^\alpha > 0. \quad (10)$$

The load displacement relation is

$$P = \sqrt{\pi}\frac{E}{1-\nu^2}\left[\frac{2}{\sqrt{\pi}}ah + a_\alpha \cdot \frac{\Gamma\left(\frac{2+\alpha}{2}\right)}{\Gamma\left(\frac{3+\alpha}{2}\right)}a^{1+\alpha}\right]. \quad (11)$$

From Equation 11, the total load has a linear relationship with the penetration depth. If the indenters have the same radius, their load–displacement curves will be parallel to each other at the complete contact region.

From Equation 11, we also have contact stiffness

$$S = \frac{\mathrm{d}P}{\mathrm{d}h} = 2\frac{E}{1-\nu^2}a. \quad (12)$$

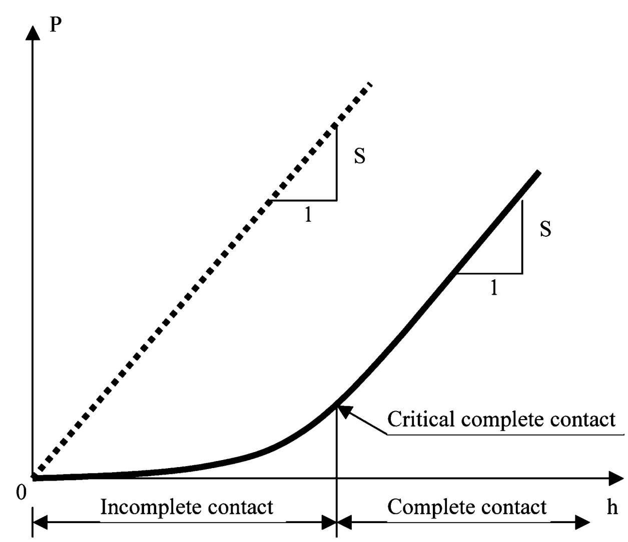

Figure 2 Load–displacement curves (solid line is for a non-flat-ended indenter, and dashed line is for a flat-ended indenter with the same radius).

Equation 12 is the same relation as the one for incomplete contact [3].

We summarize the different contact situations in Fig. 2. Before critical complete contact, the load–displacement follows a power-law relation; after that, it is a linear relationship. The whole load–displacement curve including incomplete contact and complete contact is smooth.

References

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